Some new characterizations of some old Maltsev conditions

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Outline of Talk

- Brief discussion of the Lattice of Interpretability and Maltsev conditions,
- Presentation of six special Maltsev conditions,
- Detailed discussion of these conditions, including a uniform way to present them via matrix term-conditions,
- A discussion of the locally finite case,
- Results on algorithmic questions related to testing for the Maltsev conditions.

NOTE: To go deeper into the topic of this talk, please consult the books by Hobby & McKenzie and by Kearnes & Kiss.

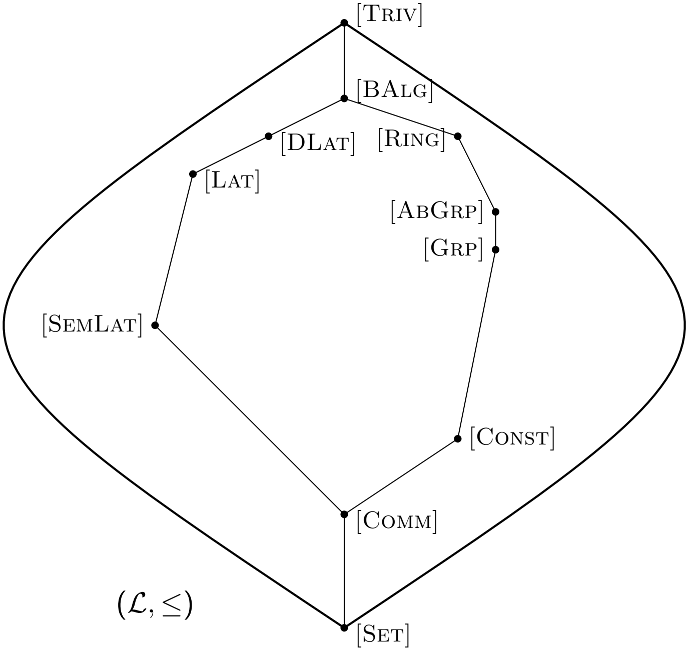
The Lattice of Interpretability

Quote from Garcia & Taylor

"In 1974, W.D. Neumann introduced an interesting lattice for comparing the relative strengths of varieties. He defined a variety $\mathcal{V}$ to be $\leq$ another variety $\mathcal{W}$ iff $\mathcal{V}$ is interpretable in $\mathcal{W}$."

Definition

- If $\mathbf{A} = \langle A, \mathcal{F} \rangle$ is an algebra and $\mathcal{S} \subseteq \mathrm{Clo}(\mathbf{A})$, then the algebra $\langle A, \mathcal{S} \rangle$ is called a term reduct of $\mathbf{A}$.
- The variety $\mathcal{V}$ is interpretable in $\mathcal{W}$ ($\mathcal{V} \leq \mathcal{W}$) if every algebra in $\mathcal{W}$ has a term reduct that is in $\mathcal{V}$.
- The order $\leq$ is a quasi-order on the class of varieties, i.e., it is reflexive and transitive.
- Modulo the equivalence relation of bi-interpretability, $\leq$ provides a complete lattice ordering on the class of varieties. This lattice is called the lattice of interpretability types of varieties.



$(\mathcal{L}, \leq)$

Maltsev Conditions

Remark

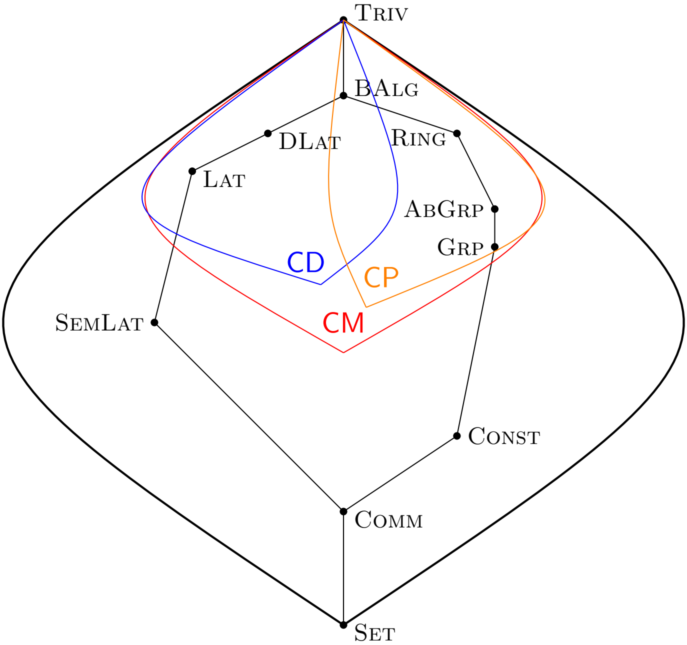
Many properties of varieties are monotone with respect to $\leq$ in the sense that if $\mathcal{V} \leq \mathcal{W}$ and $\mathcal{V}$ satisfies the property then so does $\mathcal{W}$. As such, filters in $\mathcal{L}$ are of particular importance in the study of varieties.

Definition

A Maltsev Class (or Condition) is the collection of varieties from some filter in $\mathcal{L}$ determined by a set of finitely presented varieties.

Examples

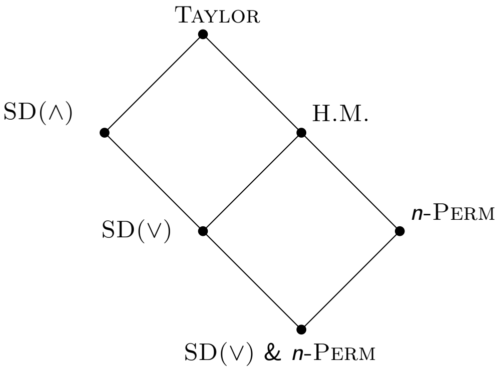
The conditions of Congruence Modularity (CM), Congruence Permutability (CP) and Congruence Distributivity (CD) are familiar examples of Maltsev Conditions.


Triv
BAlg
DLat
Ring
Lat
AbGrp
Grp
CD
CP
SemLat
CM
Const
Comm
Set

More General Maltsev Conditions

Quote from "The Structure of Finite Algebra", by Hobby & McKenzie:

"Our theory reveals a sharp division of locally finite varieties of algebras into six interesting new families, each of which is characterized by the behaviour of congruences in the algebras."



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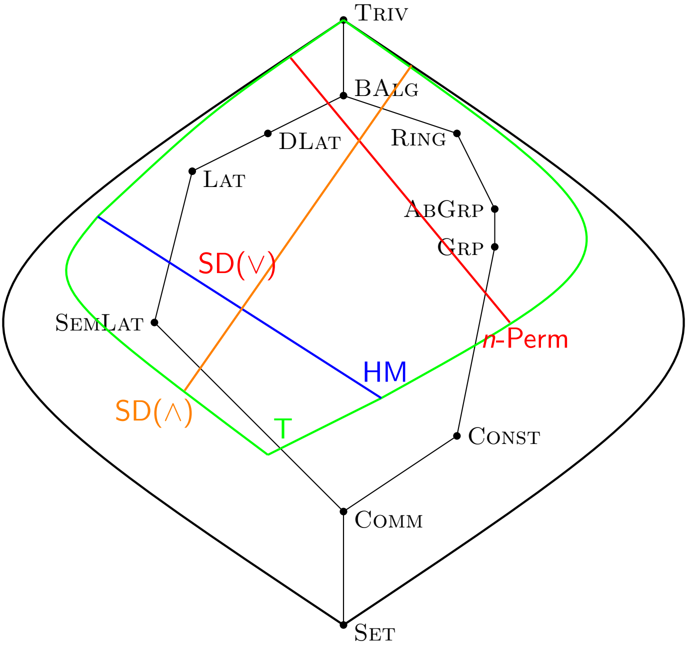
Triv
BAlg
DLat
Ring
Lat
AbGrp
Grp
SD(∨)
SemLat
n-Perm
HM
SD(∧)
T
Const
Comm
Set

Taylor Terms

Definition

- A term $t(x_1, \ldots, x_n)$ of a variety $\mathcal{V}$ is idempotent if $t(x, x, \ldots, x) \approx x$ holds in $\mathcal{V}$. $\mathcal{V}$ is idempotent if all of its term operations are.
- A Maltsev condition is idempotent if it is defined by a set of idempotent varieties.

Theorem (Taylor)

The following are equivalent for a variety $\mathcal{V}$:

- *$\mathcal{V}$ satisfies a non-trivial idempotent Maltsev condition,*
- *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in* Sets,
- *$\mathcal{V}$ has a Taylor Term.*

Taylor Terms

Definition

A term $t(x_1, \ldots, x_n)$ is a Taylor term for a variety $\mathcal{V}$ if it is idempotent and satisfies a system of n equations in the variables $\{\mathbf{x}, \mathbf{y}\}$ of the form:

$$t\begin{bmatrix} \mathbf{x} & & & \\ & \mathbf{x} & & \\ & & \ddots & \\ & & & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & & & \\ & \mathbf{y} & & \\ & & \ddots & \\ & & & \mathbf{y} \end{bmatrix}$$

Hobby-McKenzie Terms

Theorem

The following are equivalent for a variety $\mathcal{V}$:

- *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in* Semilattices,
- *$\mathcal{V}$ satisfies a non-trivial congruence identity,*
- *$\mathcal{V}$ has a Hobby-McKenzie Term.*

Definition

A term $t(x_1, \ldots, x_n)$ is a Hobby-McKenzie term for a variety $\mathcal{V}$ if it is idempotent and satisfies a system of n equations in the variables $\{\mathbf{x}, \mathbf{y}\}$ of the form:

$$t\begin{bmatrix} \mathbf{x} & & & \\ \mathbf{x} & \mathbf{x} & & \\ \mathbf{x} & \mathbf{x} & \ddots & \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & & & \\ & \mathbf{y} & & \\ & & \ddots & \\ & & & \mathbf{y} \end{bmatrix}$$

Congruence Meet Semi-Distributivity

Definition

An algebra is congruence meet semi-distributive (SD($\wedge$)) if its congruence lattice satisfies the implication:

$$\alpha \wedge \beta = \alpha \wedge \gamma \Rightarrow \alpha \wedge \beta = \alpha \wedge (\beta \vee \gamma).$$

It is congruence join semi-distributive (SD($\vee$)) if its congruence lattice satisfies the dual implication.

Theorem

The following are equivalent for a variety $\mathcal{V}$:

- *$\mathcal{V}$ is congruence meet semi-distributive,*
- *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in every non-trivial variety of modules,*
- *$\mathcal{V}$ satisfies a particular (and messy) idempotent Maltsev condition.*

Congruence Meet Semi-Distributivity

Remark

In the locally finite case, a Maltsev condition for SD($\wedge$) *varieties exists that can be expressed via a matrix term-condition.*

Theorem (Barto, Kozik)

A locally finite variety $\mathcal{V}$ *is* SD($\wedge$) *if and only if it has an idempotent term* $t(x_1, \ldots, x_n)$ *that satisfies a system of* n *equations of the form:*

$$t\begin{bmatrix} \mathbf{x} & & & \\ * & \mathbf{x} & & \\ * & * & \ddots & \\ * & * & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & & & \\ * & \mathbf{y} & & \\ * & * & \ddots & \\ * & * & \cdots & \mathbf{y} \end{bmatrix}$$

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Congruence Join Semi-Distributivity

Theorem

The following are equivalent for a variety $\mathcal{V}$:

- $\mathcal{V}$ *is congruence join semi-distributive,*
- $\mathcal{V}$ *satisfies an idempotent Maltsev condition that fails in the variety of Semilattices and in every non-trivial variety of modules.*
- *For some* k, $\mathcal{V}$ *has terms* $p_i(x, y, z)$, *for* $0 \leq i \leq k$, *which satisfy the identities:*

$$
\begin{aligned}
p_0(x, y, z) &= x \text{ and } p_k(x, y, z) = z \\
p_i(x, y, y) &= p_{i+1}(x, y, y),\ p_i(x, y, x) = p_{i+1}(x, y, x) \text{ for } i \text{ even} \\
p_i(x, x, y) &= p_{i+1}(x, x, y) \text{ for } i \text{ odd}
\end{aligned}
$$

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Congruence Join Semi-Distributivity

Theorem (Freese et al)

A variety $\mathcal{V}$ is congruence join semi-distributive if and only if it has an idempotent term $t(x_1, \ldots, x_n)$ that satisfies a system of n equations of the form:

$$t\begin{bmatrix} \mathbf{x} & & & \\ \mathbf{x} & \mathbf{x} & & \\ \mathbf{x} & \mathbf{x} & \ddots & \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & & & \\ \mathbf{x} & \mathbf{y} & & \\ \mathbf{x} & \mathbf{x} & \ddots & \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{y} \end{bmatrix}$$

Remarks

- *It is not hard to show that this term condition cannot be satisfied by any module term or any semilattice term and so it implies* SD($\vee$).
- *For the converse, using terms similar to those on the previous page, one can compose them to build a term that satisfies the above matrix condition.*

n-permutability

Remarks

- *Hagemann and Mitschke have provided a nice Maltsev condition for the class of n-permutable varieties.*
- *For locally finite varieties, Hobby and McKenzie provide alternate characterizations.*

Theorem

The following are equivalent for a locally finite variety $\mathcal{V}$:

- $\mathcal{V}$ *is congruence n-permutable for some n,*
- $\mathcal{V}$ *satisfies an idempotent Maltsev condition that fails in the variety of Distributive Lattices,*
- $\mathcal{V}$ *has terms* $p_i(x, y, z)$, *for* $0 \leq i \leq n$, *which satisfy the identities:*

$$\begin{aligned} p_0(x, y, z) &= x \text{ and } p_n(x, y, z) = z \\ p_i(x, x, y) &= p_{i+1}(x, y, y) \text{ for each } i \end{aligned}$$

Remark

By now, it should come as no surprise (assuming that you've been paying attention), that one can characterize n-permutability via a matrix term condition.

Theorem (Larose et al)

A locally finite variety $\mathcal{V}$ is congruence k-permutable for some k if and only if it has an idempotent term $t(x_1, \ldots, x_n)$, for some n, that satisfies a system of n equations of the form:

$$t\begin{bmatrix} \mathbf{x} & & & \\ \mathbf{x} & \mathbf{x} & & \\ \mathbf{x} & \mathbf{x} & \ddots & \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \\ & \mathbf{y} & \cdots & \mathbf{y} \\ & & \ddots & \mathbf{y} \\ & & & \mathbf{y} \end{bmatrix}$$

The sixth condition

Remarks

- *The strongest of the six Maltsev conditions is the conjunction of* SD(∨) *and n-permutability.*
- *So, the following matrix term condition implies this Maltsev condition, but it is not (yet) known to be equivalent to it.*

Observation

A variety $\mathcal{V}$ is SD(∨) and k-permutable for some k if it has an idempotent term $t(x_1, \ldots, x_n)$, for some n, that satisfies a system of n equations of the form:

$$t\begin{bmatrix} \mathbf{x} & & & \\ \mathbf{x} & \mathbf{x} & & \\ \mathbf{x} & \mathbf{x} & \ddots & \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & \mathbf{y} & \cdots & \mathbf{y} \\ \mathbf{x} & \mathbf{y} & \cdots & \mathbf{y} \\ \mathbf{x} & \mathbf{x} & \ddots & \mathbf{y} \\ \mathbf{x} & \mathbf{x} & \cdots & \mathbf{y} \end{bmatrix}$$

Can we do better?

Remarks

- *In the locally finite case, we might expect that better results can be obtained.*
- *Surprising results have been established for two of the Maltsev classes.*

Theorem (Siggers!!)

A locally finite variety $\mathcal{V}$ has a Taylor term if and only if it has a 4-ary term that satisfies:

$$t\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{y} \\ \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & \mathbf{y} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} \\ \mathbf{x} & \mathbf{y} & \mathbf{y} & \mathbf{y} \end{bmatrix}$$

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Can we do better?

Theorem (Freese et al)

A locally finite variety $\mathcal{V}$ *is* SD($\wedge$) *if and only if it has a 12-ary term* $t(x_1, \ldots, x_{12})$ *that satisfies a system of* 12 *equations of the form:*

$$t\begin{bmatrix} \mathbf{x} & & & \\ * & \mathbf{x} & & \\ * & * & \ddots & \\ * & * & \cdots & \mathbf{x} \end{bmatrix} \approx t\begin{bmatrix} \mathbf{y} & & & \\ * & \mathbf{y} & & \\ * & * & \ddots & \\ * & * & \cdots & \mathbf{y} \end{bmatrix}$$

Remark

It has been shown, in [Freese et al], that no such result can hold for the four other Maltsev conditions. We show that for locally finite varieties, these conditions are not equivalent to Strong Maltsev Conditions.

Algorithmic questions

General Question:

Given a finite algebra, how difficult is it to determine if the variety that it generates satisfies one of these Maltsev conditions?

Theorem (Freese, Valeriote)

For each of the six Maltsev conditions, the problem of determining if a given finite algebra generates a variety that satisfies it is an EXPTIME-complete *problem.*

Remark

It is not difficult to show that these problems are in EXPTIME, *since one can check them by constructing the 3-generated free algebra in the variety and then looking for terms that witness the given Maltsev condition.*

The Idempotent Case

Theorem (Freese, Valeriote)

For each of the six Maltsev conditions, the problem of determining if a given finite idempotent algebra generates a variety that satisfies it is in P.

Remarks

- *For the Taylor-term class, this was established by Bulatov.*
- *Using results from Hobby-McKenzie, we generalize Bulatov's approach to construct poly-time algorithms to solve the other 5 cases.*
- *We also construct poly-time algorithms to test for Congruence Permutabilty, Distributivity, and Modularity in the idempotent case.*

The Relational Case

Remarks

- *Of particular interest are the related questions for relational structures, namely,*
- *Given a relational structure* **B**, *how difficult is it to determine if the algebra of polymorphisms of* **B** *generates a variety that satisfies one of the six Maltsev conditions?*
- *These problems are known to be decidable, and in some cases, there are better results.*

Results:

- Testing for a Taylor term is in NP. (This follows from Siggers's result.)
- [Bulatov] Testing for SD($\wedge$) is in P.
- More generally, Maroti has observed that any special Maltsev condition that implies SD($\wedge$) can be settled in polynomially time.

Open Problems

- Verify that the matrix condition for the sixth Maltsev class in our list characterizes the (locally finite) varieties in this class.
- Determine if all of the given matrix term conditions work in the non-locally finite case. [NOTE: Since this talk was given, it has been shown that the matrix term-condition given for n-permutability works in the non-locally finite case as well.]
- Show that if a variety $\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in Distributive Lattices, then $\mathcal{V}$ is congruence n-permutable for some n.
- Resolve the open computational complexity questions for relational structures.
- Find the complexity of determining if a given finite algebra has a Maltsev term or a majority term.