Relational structures, Maltsev conditions, and CSP

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9 June 2011

Outline

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1. Motivating examples

Example #1: Structures supporting Maltsev conditions

(Barto, Kozik, Niven, *SIAM J. Comput.*, 2009) – Smooth digraphs

Recall:

- **Digraph**: a set V with binary relation $E \subseteq V \times V$.
- Digraph (V, E) is **smooth** if $\forall y \in V \ \exists x, z \in V$ with $(x, y), (y, z) \in E$.
- **WNU operation**: a k-ary function f $(k \geqslant 2)$ satisfying

$$f(y, x, x, \ldots, x) \approx f(x, y, x, \ldots, x) \approx \cdots \approx f(x, x, x, \ldots, y)$$

$$\text{and} \quad f(x, x, x, \ldots, x) \approx x \quad (\text{"idempotence"}).$$

Theorem (Barto, Kozik, Niven)

Every finite smooth digraph admitting a compatible WNU operation has "nice" structure.

Example #2: Structures realized in algebras

(Siggers, *Alg. Univ.*, 2010) – Siggers terms

Definition

A **Siggers operation** is an idempotent 6-ary operation $s(x_1, \dots, x_6)$ satisfying

$$\begin{aligned} s(x,x,x,x,y,y) &\approx s(x,y,x,y,x,x) \\ s(y,y,x,x,x,x) &\approx s(x,x,y,x,y,x). \end{aligned}$$

Siggers' Theorem: if algebra $\mathbf{A}$ is finite and $V(\mathbf{A})$ "omits type 1," then $\mathbf{A}$ has a Siggers term operation.

Key step (Siggers)

If $\mathbf{A}$ is idempotent, then **A fails** to have a Siggers term operation $\Leftrightarrow$ there exists a (simple) graph $\mathbb{G}$ containing a triangle which can be "realized" in a member of $V(\mathbf{A})$.

Example #3: Structures definable in other structures

(Nešetřil, Siggers, Zadorí, *Euro. J. Combin* 2010) – CSP Dichotomy

Background: Let $\mathbb{H}$ be an arbitrary finite relational structure.

- $\mathrm{CSP}(\mathbb{H})$ is a combinatorial decision problem, depending on $\mathbb{H}$.
- $\mathrm{core}(\mathbb{H})$ is the unique (up to $\cong$) minimal retract of $\mathbb{H}$.
- $\mathbb{H}^c$ is the structure which results from adding to $\mathbb{H}$ all the singleton unary relations $\{a\}$ ($a \in H$). (Called "$\mathbb{H}$ **with constants**.")
- BJK is the class of $\mathbb{H}$ for which Bulatov, Jeavons and Krokhin conjecture $\mathrm{CSP}(\mathbb{H})$ should be NP-complete.

Theorem (Nešetřil, Siggers, Zadorí)

Assume $\mathrm{core}(\mathbb{H}) = \mathbb{H}$. *Then* $\mathbb{H} \in \mathrm{BJK} \Leftrightarrow$ *there exists a graph* $\mathbb{G}$ *whose core is* $\mathbb{K}_3$ *such that* $\mathbb{G}$ *is "pp-definable" in* $\mathbb{H}^c$.

In this lecture I propose a possible general framework for discussing finite relational structures from the point of view of:

- the (strong) Maltsev conditions they support,
- their realizations in algebras and varieties, and
- their definability within each other.

2. Many definitions

1. **Finite relational structure**: $\mathbb{H} = (H; R_1, R_2, \ldots)$ where
   - H is a finite set (the **domain**, or **universe**, or **underlying set**);
   - $R_1, R_2, \ldots$ is a list (possibly infinite) of finitary relations on H.
   - Note: H and each R_i always assumed to be **nonempty**.

2. **Polymorphism** of $\mathbb{H}$: any finitary operation f on H which **preserves** (or **is compatible with**) each relation R_i.
   - Equivalently, any homomorphism $f : \mathbb{H}^k \to \mathbb{H}$ $(k \geqslant 1)$.

3. The **polymorphism algebra of** $\mathbb{H}$, denoted $\mathrm{alg}(\mathbb{H})$, is the algebra with universe H and set of operations $= \mathrm{Pol}(\mathbb{H}) := \{\text{all polymorphisms of } \mathbb{H}\}$.

4. **Strong Maltsev condition**: any finite set of identities.

Example:

$$\Sigma_{maj} = \{m(x,x,y) \approx x,\ m(x,y,x) \approx x,\ m(y,x,x) \approx x\}.$$

5. Let $\mathbb{H}$ be a finite relational structure.
   Let Σ be a strong Maltsev condition.

Definition

$\mathbb{H}$ **supports** Σ iff $V(\mathrm{alg}(\mathbb{H}))$ satisfies Σ in the usual way.

- Informally, iff there exist polymorphisms of $\mathbb{H}$ which make the identities in Σ true.

Example. $\mathbb{H}$ supports Σ_{maj} iff $\mathbb{H}$ has a majority polymorphism.

6. **Realizations of structures in algebras**

Let $\mathbb{H} = (H; R_1, R_2, \ldots)$ be a finite relational structure with $\text{arity}(R_i) = n_i$ for $i = 1, 2, \ldots$.

Let $\mathbf{A}$ be a finite algebra.

Definition

$\mathbb{H}$ is **realized** in $\mathbf{A}$ iff for $i = 1, 2, \ldots$ there exist $\mathbf{B}_i \leqslant \mathbf{A}^{n_i}$ such that

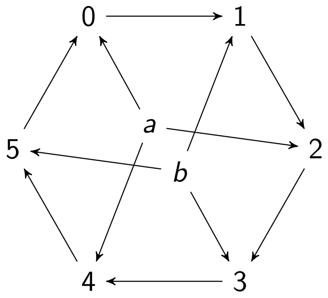
$$\mathbb{H} \;\cong\; (A; B_1, B_2, \ldots).$$

Example: Let $\mathbf{A} = (2; m)$ where $2 = \{0, 1\}$ and m is the (unique) majority operation on 2. Let $\mathbb{H}$ be the graph $\mathbb{K}_2 = \;(\bullet \rightleftarrows \bullet)$.

$\mathbb{K}_2$ is realized in $\mathbf{A}$, since $\mathbf{B} := \{(0, 1), (1, 0)\} \leqslant \mathbf{A}^2$ and $\mathbb{K}_2 \cong (2; B)$.

7. **Pp-definability of structures** – first an example

Let $\mathbb{H} = (H; \to)$ be the following directed graph:



I want to "define" the graph $\mathbb{K}_2$ in $\mathbb{H}$.

In $\mathbb{H}$, the formula $\upsilon(x)$ given by $\exists z[z \to x]$ defines the unary relation

$$U = \{0, 1, 2, 3, 4, 5\}.$$

Similarly, the formula $\vartheta(x, y)$ given by

$$\exists z_1, \ldots, z_5[z_1 \to x \ \wedge \ z_1 \to z_2 \to z_3 \to z_4 \to z_5 \to y]\}$$

defines the binary relation

$$\Theta = \{0, 2, 4\}^2 \cup \{1, 3, 5\}^2,$$

which is an equivalence relation on U with two classes.

Finally, the formula $\exists z[\vartheta(x, z) \ \wedge \ x \to y]$ defines the relation

$$E = (\{0, 2, 4\} \times \{1, 3, 5\}) \cup (\{1, 3, 5\} \times \{0, 2, 4\}).$$

As $(U/\Theta; E/\Theta) \cong \mathbb{K}_2$, we have "defined" $\mathbb{K}_2$ in $\mathbb{H}$.

In general, we will use formulas of the kind above ("pp-formulas"), allowing tuples in place of individual variables.

Definition

A **primitive positive** (or **pp-**) **formula** is a first-order formula built from atomic formulas (basic relations and $=$) using only $\wedge$ and $\exists$.

Fix a finite relational structure $\mathbb{H} = (H; R_1, R_2, \ldots)$.

Definition

A relation $S \subseteq H^n$ is **pp-definable** in $\mathbb{H}$ iff there exists a pp-formula in the language of $\mathbb{H}$, with n free variables, whose set of solutions in $\mathbb{H}$ is S.

- (Equivalently, iff S belongs to the "relational clone" generated by $\{R_1, R_2, \ldots\}$.)

Let Θ be an equivalence relation on H. The **quotient** $\mathbb{H}/\Theta$ is

$$\mathbb{H}/\Theta \;=\; (H/\Theta; R_1/\Theta, R_2/\Theta, \ldots)$$

where $R_i/\Theta = \{(a_1/\Theta, \ldots, a_{n_i}/\Theta) : (a_1, \ldots, a_{n_i}) \in R_i\}$.

Finally, the general definition.

Let $\mathbb{G}, \mathbb{H}$ be finite relational structures.
Write $\mathbb{G} = (G; R_1, R_2, \ldots)$ with $\text{arity}(R_i) = n_i$.

Definition

$\mathbb{G}$ is **pp-definable in** $\mathbb{H}$ iff there exist:

- $k \geqslant 1$
- Pp-definable relations of $\mathbb{H}$:
  - $U \subseteq H^k$
  - $\Theta \subseteq U^2 \quad (\subseteq (H^k)^2 = H^{2k})$
  - $S_i \subseteq U^{n_i} \quad (\subseteq (H^k)^{n_i} = H^{n_i k})$ for $i = 1, 2, \ldots$

such that

- Θ is an equivalence relation on U.
- $\mathbb{G} \cong (U; S_1, S_2, \ldots)/\Theta$.

Notation: $\mathbb{G} \leqslant_{pp} \mathbb{H}$.

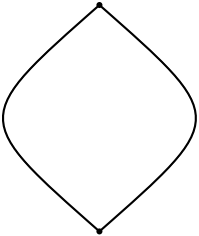
Remarks.

Let $\mathcal{R} := \{\text{all finite relational structures}\}$.

1. $\leqslant_{pp}$ is a quasi-order (reflexive and transitive) on $\mathcal{R}$ ...
2. ... so induces an equivalence relation on $\mathcal{R}$ ...
   (**Notation**: $[\mathbb{H}] = \{\mathbb{G} : \mathbb{H} \leqslant_{pp} \mathbb{G} \leqslant_{pp} \mathbb{H}\}$)
3. ... and a partial ordering on the set of equivalence classes.

Notation: $\mathcal{L}_{pp}$ = the poset $\{[\mathbb{H}] : \mathbb{H} \in \mathcal{R}\}$ of equivalence classes ordered by $\leqslant_{pp}$.

$\mathcal{L}_{pp} =$



$[\mathbf{1}]$

3. A theorem

Connecting the pieces:

Theorem

Suppose $\mathbb{G}, \mathbb{H}$ are finite relational structures. The following are equivalent:

1. $\mathbb{G}$ *supports every strong Maltsev condition supported by* $\mathbb{H}$.
2. $\mathbb{G}$ *is realized in some member of* $V(\mathrm{alg}(\mathbb{H}))$.
3. $\mathbb{G} \leqslant_{pp} \mathbb{H}$.

Proof sketch: Write $\mathbb{G} = (G; R_1, R_2, \ldots)$. Let $\mathbf{G} = \mathrm{alg}(\mathbb{G})$, $\mathbf{H} = \mathrm{alg}(\mathbb{H})$.

1. $\mathbb{G}$ supports every strong Maltsev condition supported by $\mathbb{H}$.
2. $\mathbb{G}$ is realized in some member of $V(\mathbf{H})$.
3. $\mathbb{G} \leqslant_{pp} \mathbb{H}$.

(3) $\Rightarrow$ (2). Assume $\mathbb{G}$ is pp-defined in $\mathbb{H}$ via $k \geqslant 1$ and $U, \Theta, S_1, S_2, \ldots$.

U, Θ pp-definable in $\mathbb{H}$ implies $\mathbf{U} \leqslant \mathbf{H}^k$ and $\Theta \in \mathrm{Con}(\mathbf{U})$. Let $\mathbf{A} = \mathbf{U}/\Theta$.

The S_i can be similarly used to produce $\mathbf{B}_i \leqslant \mathbf{A}^{n_i}$ so that

$$(A; B_1, B_2, \ldots) \;=\; (U; S_1, S_2, \ldots)/\Theta \;\cong\; \mathbb{G}.$$

(2) $\Rightarrow$ (1). Assume $\mathbb{G}$ is realized in $\mathbf{A} \in V(\mathbf{H})$.

This implies $\mathbf{A}$ is isomorphic to a reduct of $\mathbf{G}$.

Assume Σ is a strong Maltsev condition supported by $\mathbb{H}$, i.e., $V(\mathbf{H})$ satisfies Σ.

In particular, $\mathbf{A}$ satisfies Σ, hence so must $\mathbf{G}$, i.e., $\mathbb{G}$ supports Σ.

(Proof sketch continued.)

(Recall $\mathbb{G} = (G; R_1, R_2, \ldots)$, $\mathbf{G} = \mathrm{alg}(\mathbb{G})$, $\mathbf{H} = \mathrm{alg}(\mathbb{H})$.)

(1) $\Rightarrow$ (3). Requires more work.

Assume $\mathbb{G}$ supports every strong Maltsev condition supported by $\mathbb{H}$.
(Must show $\mathbb{G}$ is pp-definable in $\mathbb{H}$.)

By a compactness argument, may assume that the signature of $\mathbb{G}$ is finite.

Let N be large enough.[1]

Let $\Sigma_{\mathbb{H},N}$ denote the strong Maltsev condition which describes all compositions among $\mathrm{Pol}_{(\leqslant N)}(\mathbb{H})$, the at-most-$N$-ary fragment of $\mathrm{Pol}(\mathbb{H})$.

By assumption, $\mathbb{G}$ supports $\Sigma_{\mathbb{H},N}$. This gives a clone homomorphism $\mathrm{Pol}_{(\leqslant N)}(\mathbb{H}) \to \mathrm{Pol}_{(\leqslant N)}(\mathbb{G})$, which I denote $s \mapsto s^\alpha$.

[1] $N > |G|$, $N \geqslant |R_i|$ for all i.

(Proof of (1) $\Rightarrow$ (3), continued)

Let $n = |G|$ and $\mathbf{F} = \mathbf{F}_{V(\mathbf{H})}(n)$, canonically with universe $F = \mathrm{Pol}_n(\mathbb{H})$.

Fix an enumeration $G = \{a_1, \ldots, a_n\}$.

Key: define $\beta : F \to G$ by $\beta(s) = s^{\alpha}(a_1, \ldots, a_n)$.

Show that $\Theta := \ker(\beta) \in \mathrm{Con}(\mathbf{F})$, and that each Θ-block contains exactly one projection.

Use β^{-1} to lift each $R_i \leqslant \mathbf{G}^{n_i}$ to $S_i \subseteq F^{n_i}$.

Show that each S_i is a subuniverse of $\mathbf{F}^{n_i}$ (hence of $(\mathbf{H}^{H^n})^{n_i}$).

$F, \Theta, S_1, S_2, \ldots$ witness $\mathbb{G}$ being pp-definable in $\mathbb{H}$. □

4. Applications

Application #1. $[\mathbb{K}_3]$ is the "top" element of $\mathcal{L}_{pp}$.

Equivalently, *every* finite relational structure is pp-definable in $\mathbb{K}_3$.

Proof. It suffices by the previous theorem to show that every strong Maltsev condition supported by $\mathbb{K}_3$ is trivial (i.e., supported by all finite structures).

This can be proved directly (and easily), modulo the following fact:

$\mathbb{K}_3$ is **projective**, i.e., core and every polymorphism depends on only one variable. □

Amusing exercise: find an explicit pp-definition of $(\mathbb{K}_3)^c$ in $\mathbb{K}_3$.

Application #2. If $\mathbb{T}$ is a finite simple graph which is *not* bipartite, then $\mathbb{T}$ supports no nontrivial *idempotent* Maltsev condition.

Equivalently, $[\mathbb{T}^c] =$ "top" element of $\mathcal{L}_{pp}$.

What is the simplest proof?

Here is a proof based on Bulatov's re-proof of the Hell-Nešetřil theorem (*Theor. Comp. Sci.* 2005).

Bulatov starts by assuming $\mathbb{T}$ "to be the smallest [graph] amongst all non-bipartite graphs that can be derived from" $\mathbb{T}$.

From this assumption he argues that $\mathbb{T} = \mathbb{K}_3$.

By examining his argument carefully, one sees that it works assuming only that if $\mathbb{G}$ is a finite graph and $\mathbb{G} \leqslant_{pp} \mathbb{T}^c$, then $\mathbb{G}$ is "derivable" from $\mathbb{T}$. [2]

[2] Though in the first step Bulatov assumes that $\mathbb{T}$ is core, this part of his argument never actually uses this assumption.

Thus Bulatov's argument establishes the following:

Fact: if $\mathbb{T}$ is a finite non-bipartite graph, then there exists a sequence $\mathbb{G}_0, \mathbb{G}_1, \ldots, \mathbb{G}_n$ of finite non-bipartite graphs such that

- $\mathbb{G}_0 = \mathbb{T}$.
- $\mathbb{G}_n = \mathbb{K}_3$.
- For each $i < n$, $\mathbb{G}_{i+1} \leqslant_{pp} (\mathbb{G}_i)^c$.

It is easy to check that $\mathbb{G} \leqslant_{pp} \mathbb{H}^c$ implies $\mathbb{G}^c \leqslant_{pp} \mathbb{H}^c$.

This, with transitivity of $\leqslant_{pp}$, gives $\mathbb{K}_3 \leqslant_{pp} \mathbb{T}^c$.

As $[\mathbb{K}_3] =$ "top," this implies $[\mathbb{T}^c] =$ "top." □

Application #3: Proof of Siggers' theorem.

Recall the key construction:

If $\mathbb{H}$ has no Siggers polymorphism, then there exists a graph $\mathbb{T}$ containing a triangle such that $\mathbb{T} \leqslant_{pp} \mathbb{H}^c$.

Remaining step: Show that such $\mathbb{T}$ cannot support any idempotent Maltsev condition. (Hence neither can $\mathbb{H}$.) How to show it?

Could cite Barto-Kozik-Niven. Siggers cited Bulatov's 2005 paper.

Now we see that his citation is correct(ed): if $\mathbb{T}$ contains a triangle, then it is not bipartite. Hence $\mathbb{K}_3 \leqslant_{pp} \mathbb{T}^c$ (Application #2).

$\mathbb{T} \leqslant_{pp} \mathbb{H}^c$ implies $\mathbb{T}^c \leqslant_{pp} \mathbb{H}^c$.

Hence $\mathbb{K}_3 \leqslant_{pp} \mathbb{H}^c$ by transitivity of $\leqslant_{pp}$. □

Application #4: stating the Algebraic CSP Dichotomy Conjecture

Let $\mathbf{2}_{\mathbf{NAE}} = (2; R)$ where $R = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$.

$\mathbf{2}_{\mathbf{NAE}}$ is projective, so $[\mathbf{2}_{\mathbf{NAE}}] = [\mathbf{2}_{\mathbf{NAE}}{}^c] =$ "top" element of $\mathcal{L}_{pp}$.

The class BJK of those core $\mathbb{H}$ (with finite signature) for which $\mathrm{CSP}(\mathbb{H})$ is conjectured to be NP-complete is normally characterized by either of the equivalent conditions:

1. $\mathrm{alg}(\mathbb{H}^c)$ satisfies no nontrivial (idempotent) Maltsev condition.
2. $\mathbf{2}_{\mathbf{NAE}}{}^c$ is realized in $V(\mathrm{alg}(\mathbb{H}^c))$ [or in $HS(\mathrm{alg}(\mathbb{H}^c))$].

To these we can add

3. $\mathbb{K}_3 \leqslant_{pp} \mathbb{H}^c$.
4. (Nešetřil, Siggers, Zadorí) $\mathbb{G} \leqslant_{pp} \mathbb{H}^c$ for some finite graph $\mathbb{G}$ whose core is $\mathbb{K}_3$. ("$\mathbb{K}_3$-partitionability.")
5. $[\mathbb{H}^c] =$ "top" element of $\mathcal{L}_{pp}$.

5. Problems

1. Suppose $\mathbb{H}$ is core and $[\mathbb{H}^c]_{pp} = top$. Does this imply $[\mathbb{H}]_{pp} = top$? (Yes if $\mathbb{H}$ is also projective.)

2. Does there exist a strong Maltsev condition Σ such that, $\forall\, \mathbb{H} \in \mathcal{R}$, $\mathbb{H}$ supports Σ iff $[\mathbb{H}]_{pp} \neq top$?

3. Find a "constructive" characterization of this binary relation on strong Maltsev conditions: "Γ is supported by all $\mathbb{H} \in \mathcal{R}$ which support Σ."

4. The relation "$\mathbb{H}$ supports Σ" induces a Galois connection between the subsets of $\mathcal{R}$ and the subsets of the set of all strong Maltsev conditions.
   Characterize the closure operator on the relational structure side of this Galois connection.

5. Ditto for the strong Maltsev condition side.