Admissible Rules in Logic and Algebra

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An Observation of Andrzej Wroński

The following rule is **admissible** but not **derivable** in the implication-negation fragment of intuitionistic logic IPC:

$$p \to \neg q,\ (\neg\neg p \to p) \to r,\ (\neg\neg q \to q) \to r \ \ / \ \ r.$$

I.e., this fragment is not **structurally complete**.

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However, the implication and implication-conjunction-negation fragments of IPC *are* structurally complete…

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Prompting the Questions. . .

When structural completeness *fails*,

- is the set of admissible rules **decidable**?
- what is its **complexity**?
- is there some elegant (finite) **axiomatization**?
- how do these properties relate to **other logics**?
- can the admissible rules be **useful**?

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Overview

This talk will expand on

- the role of admissible rules in logic and algebra
- the intriguing case of the implication-negation fragment of IPC
- some interesting applications of admissible rules.

Some Terminology

To talk about logics and algebras, we need

- **propositional languages** $\mathcal{L}$ consisting of connectives such as $\wedge, \vee, \to, \neg, \bot, \top$ with specified finite arities
- finite sets (denoted Γ, Δ) of $\mathcal{L}$-**formulas** (denoted ψ, φ, χ) built from a countably infinite set of variables (denoted p, q, r)
- endomorphisms on $\mathbf{Fm}_{\mathcal{L}}$ called $\mathcal{L}$-**substitutions**.

Definition

An $\mathcal{L}$-**rule** is an ordered pair (Γ, Δ), written $\Gamma \;/\; \Delta$, where $\Gamma \cup \Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ is *finite*; the rule is called **single-conclusion** if $|\Delta| = 1$.

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Logics and Consequence

Definition

A **logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of single-conclusion $\mathcal{L}$-rules satisfying (writing $\Gamma \vdash_{\mathrm{L}} \varphi$ for $(\Gamma, \{\varphi\}) \in \mathrm{L}$):

- $\{\varphi\} \vdash_{\mathrm{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma \cup \Gamma' \vdash_{\mathrm{L}} \varphi$ (monotonicity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi$, then $\Gamma \vdash_{\mathrm{L}} \psi$ (transitivity)
- if $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathrm{L}} \sigma\varphi$ for any $\mathcal{L}$-substitution σ (structurality).

An **L-theorem** is a formula φ such that $\emptyset \vdash_{\mathrm{L}} \varphi$ (abbreviated as $\vdash_{\mathrm{L}} \varphi$).

Multiple-Conclusion Consequence

Definition

An **m-logic** L on $\mathbf{Fm}_{\mathcal{L}}$ is a set of $\mathcal{L}$-rules (writing $\Gamma \vdash_{\mathsf{L}} \Delta$ for $(\Gamma, \Delta) \in \mathsf{L}$) satisfying:

- $\{\varphi\} \vdash_{\mathsf{L}} \varphi$ (reflexivity)
- if $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\Gamma \cup \Gamma' \vdash_{\mathsf{L}} \Delta' \cup \Delta$ (monotonicity)
- if $\Gamma \vdash_{\mathsf{L}} \{\varphi\} \cup \Delta$ and $\Gamma \cup \{\varphi\} \vdash_{\mathsf{L}} \Delta$, then $\Gamma \vdash_{\mathsf{L}} \Delta$ (transitivity)
- if $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\sigma\Gamma \vdash_{\mathsf{L}} \sigma\Delta$ for each $\mathcal{L}$-substitution σ (structurality).

Derivable and Admissible Rules

Definition

For a logic L on $\mathbf{Fm}_{\mathcal{L}}$, an $\mathcal{L}$-rule $\Gamma \;/\; \Delta$ is

- **L-derivable**, written $\Gamma \vdash_{\mathrm{L}} \Delta$, if $\Gamma \vdash_{\mathrm{L}} \varphi$ for some $\varphi \in \Delta$.
- **L-admissible**, written $\Gamma \mathrel{|\!\!\sim}_{\mathrm{L}} \Delta$, if for every $\mathcal{L}$-substitution σ:

$$\vdash_{\mathrm{L}} \sigma\varphi \;\text{ for all } \varphi \in \Gamma \quad \Rightarrow \quad \vdash_{\mathrm{L}} \sigma\psi \;\text{ for some } \psi \in \Delta.$$

(Note: $\vdash_{\mathrm{L}}$ and $\mathrel{|\!\!\sim}_{\mathrm{L}}$ are m-logics.)

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Structural and Universal Completeness

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Examples

- For **intuitionistic logic** IPC, the "independence of premises" rule

$$\neg p \to (q \lor r) \ / \ (\neg p \to q) \lor (\neg p \to r)$$

is admissible but non-derivable, also the "disjunction property"

$$p \lor q \ / \ p, q.$$

- For the **relevant logics** R and RM, the "disjunctive syllogism" rule

$$\neg p, p \lor q \ / \ q$$

is admissible but not derivable.

- For **Łukasiewicz infinite-valued logic Ł**, the following rules are admissible and non-derivable

$$p \leftrightarrow \neg p \ / \ \bot \qquad \text{and} \qquad p \lor \neg p \ / \ p, \neg p.$$

Examples

- For **intuitionistic logic** IPC, the "independence of premises" rule

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Unification and Admissibility

Definition

An **L-unifier** of $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is a substitution σ such that $\vdash_{\mathsf{L}} \sigma\varphi$ for all $\varphi \in \Gamma$.

On the one hand:

$$\Gamma \text{ is } \mathsf{L}\text{-unifiable} \quad \Longleftrightarrow \quad \Gamma \;/\; \emptyset \;\text{ is not } \mathsf{L}\text{-admissible.}$$

But also, if $\mathcal{C}$ is a **complete set of L-unifiers** for Γ (i.e., any L-unifier σ of Γ is less general than some $\sigma' \in \mathcal{C}$), then

$$\Gamma \;/\; \Delta \;\text{ is } \mathsf{L}\text{-admissible} \quad \Longleftrightarrow \quad \text{for all } \sigma \in \mathcal{C},\; \vdash_{\mathsf{L}} \sigma\varphi \;\text{ for some } \varphi \in \Delta.$$

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The Algebraic Perspective (1)

Quasi-equations are expressions of the form

$$\Sigma \ \Rightarrow \ \varphi \approx \psi \quad \text{interpreted as} \ \bigwedge \Sigma \ \Rightarrow \ \varphi \approx \psi$$

where Σ is a finite set of equations.

A **quasivariety** $\mathcal{Q}$ is a class of algebras of the same language satisfying some set of quasi-equations.

A **universal class** $\mathcal{U}$ is a class of algebras of the same language satisfying some set of universal formulas of the form

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The Algebraic Perspective (2)

For a class of algebras $\mathcal{K}$ of the same language:

- $\mathbb{V}(\mathcal{K})$ denotes the **variety generated by** $\mathcal{K}$
- $\mathbb{Q}(\mathcal{K})$ denotes the **quasivariety generated by** $\mathcal{K}$
- $\mathbb{U}(\mathcal{K})$ denotes the **universal class generated by** $\mathcal{K}$.
- $\mathbb{U}^+(\mathcal{K})$ denotes the **positive universal class generated by** $\mathcal{K}$.

The Algebraic Perspective (3)

Let $\mathbf{F}_{\mathcal{Q}}$ denote the **free algebra on countably many generators** of a quasivariety $\mathcal{Q}$.

Definition

$\mathcal{Q}$ is **structurally complete** if $\mathcal{Q} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}})$. (Or, any proper subquasivariety of $\mathcal{Q}$ generates a proper subvariety of $\mathbb{V}(\mathcal{Q})$.)

Definition

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An **algebraizable logic** L is structurally (universally) complete if and only if its equivalent quasivariety is structurally (universally) complete.

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An **algebraizable logic** L is structurally (universally) complete if and only if its equivalent quasivariety is structurally (universally) complete.

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Examples

- The variety of **lattices** is not structurally complete; e.g., the semi-distributivity property

$$p \wedge q \approx p \wedge r \;\Rightarrow\; p \wedge q \approx p \wedge (q \vee r)$$

holds in free lattices, but not all lattices, also Whitman's condition

$$p \wedge q \leq r \vee s \;\Rightarrow\; p \leq r \vee s,\ q \leq r \vee s,\ p \wedge q \leq q,\ p \wedge q \leq s.$$

- The variety of **abelian groups** is not structurally complete; e.g.,

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A Selective Survey

Class of algebras	Structurally complete?
Boolean algebras	yes
Heyting algebras	no
Hilbert algebras	yes
MV-algebras	no
Komori's C algebras	yes
abelian groups	no
lattice-ordered abelian groups	yes
lattices	no
distributive lattices	yes
Sugihara monoids	no
positive Sugihara monoids	yes
BCI algebras	no
BCK algebras	no

Axiomatizing Admissible Rules

For a logic L, we seek a set of rules that "axiomatizes" $\mathrel{|\!\sim}_{\mathrm{L}}$ over $\vdash_{\mathrm{L}}$ (or for a quasivariety $\mathcal{Q}$, universal formulas that axiomatize $\mathbb{U}(\mathbf{F}_{\mathcal{Q}})$).

Definition

A **basis** for $\mathrel{|\!\sim}_{\mathrm{L}}$ over L is a set B of rules such that $\mathrel{|\!\sim}_{\mathrm{L}}$ is the smallest m-logic extending $B \cup \mathrm{L}$.

We may also consider bases for "single-conclusion" $\mathrel{|\!\sim}_{\mathrm{L}}$ (or for a quasivariety $\mathcal{Q}$, quasi-equations that axiomatize $\mathbb{Q}(\mathbf{F}_{\mathcal{Q}})$ over $\mathcal{Q}$).

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Intuitionistic Logic and the Visser Rules

Iemhoff and Rozière established independently that the "Visser rules"

$$\bigwedge_{i=1}^{n}(p_i \to q_i) \to (p_{n+1} \vee p_{n+2}) \ / \ \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(p_i \to q_i) \to p_j)$$

for $n = 2, 3, \ldots$ together with the disjunction property provide a basis for the admissible rules of **intuitionistic logic**.

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(Weak) Projectivity

Suppose that the logic L has an implication connective $\to$ that admits modus ponens: $\varphi, \varphi \to \psi \vdash_{\mathsf{L}} \psi$.

Definition

$\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ is **L-projective** if for some L-unifier σ for Γ:

$$\Gamma \vdash_{\mathsf{L}} \sigma\varphi \to \varphi \quad \text{and} \quad \Gamma \vdash_{\mathsf{L}} \varphi \to \sigma\varphi \quad \text{for all } \varphi \in \mathrm{Fm}_{\mathcal{L}}.$$

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If Γ is weakly L*-projective, then $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$ if and only if $\Gamma \vdash_{\mathsf{L}} \Delta$.*

Proof.

($\Leftarrow$) If $\Gamma \vdash_{\mathsf{L}} \Delta$, then $\Gamma \vdash_{\mathsf{L}} \varphi$ for some $\varphi \in \Delta$. So $\sigma\Gamma \vdash_{\mathsf{L}} \sigma\varphi$ for any substitution σ, and if $\vdash_{\mathsf{L}} \sigma\psi$ for each $\psi \in \Gamma$, then $\vdash_{\mathsf{L}} \sigma\varphi$. I.e., $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$.

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Fragments of IPC

Theorem

The implication-conjunction fragment of IPC *is universally complete.*

Proof.

Each implication-conjunction formula φ has an IPC-projective unifier defined by $\sigma p = \varphi \to p$. We prove (by induction on formula complexity) that $\vdash_{\mathrm{IPC}} \sigma\psi \to (\varphi \to \psi)$ and $\vdash_{\mathrm{IPC}} (\varphi \to \psi) \to \sigma\psi$ for all implication-conjunction formulas ψ. It then follows easily that $\varphi \vdash_{\mathrm{IPC}} \psi \to \sigma\psi$ and $\varphi \vdash_{\mathrm{IPC}} \sigma\psi \to \psi$ for all such formulas and $\vdash_{\mathrm{IPC}} \sigma\varphi$. $\square$

Indeed, the $\{\to\}$, $\{\to, \wedge\}$, $\{\to, \wedge, \neg\}$ fragments of *all* intermediate logics are universally complete. But what of the $\{\to, \neg\}$ fragments?

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Preliminaries

Let L be the implication-negation fragment of some intermediate logic (consistent axiomatic extension of IPC) with defined constants $\top = q \to q$ for some variable q and $\bot = \neg\top$.

We will write $\vec{p} \to q$ for $p_1 \to (p_2 \to (\ldots \to (p_n \to q) \ldots))$.

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The Wroński Rules

We consider the following "Wroński rules" ($n \in \mathbb{N}$):

$$(\mathrm{W}_n) \quad (\vec{p} \to \bot) \;/\; (\neg\neg p_1 \to p_1), \ldots, (\neg\neg p_n \to p_n).$$

Lemma

(W_n) *is* Ł*-admissible for each* $n \in \mathbb{N}$.

Proof.

Suppose that $\vdash_{Ł} \sigma(\vec{p} \to \bot)$. Then $\sigma(p_i) = \vec{\varphi} \to \bot$ for some $i \in \{1, \ldots, n\}$ (otherwise, $\sigma'(q) = \top$ for each variable q gives $\vdash_{Ł} \sigma'\sigma(\vec{p} \to \bot)$ and $\vdash_{Ł} \top \to \bot$). Hence, $\vdash_{Ł} \sigma(\neg\neg p_i \to p_i)$. □

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We consider the following “Wroński rules” ($n \in \mathbb{N}$):

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Completeness

Theorem

$\mathsf{W} = \{(\mathsf{W}_n) \mid n \in \mathbb{N}\}$ *is a basis for the admissible rules of* L.

Proof.

(Sketch) Call a formula having one of the following forms *simple*:

(i) $\vec{p} \to \bot$; (ii) $\vec{\psi} \to r$ where each ψ_i has the form $p \to q$ or p.

Given a rule $\Gamma \,/\, \Delta$, find a set Γ' of simple formulas such that:

$$\Gamma \mathrel{|\!\sim}_{\mathsf{L}} \Delta \quad \Rightarrow \quad \Gamma' \mathrel{|\!\sim}_{\mathsf{L}} \Delta \qquad \text{and} \qquad \Gamma' \vdash_{\mathsf{L+W}} \Delta \quad \Rightarrow \quad \Gamma \vdash_{\mathsf{L+W}} \Delta.$$

Apply the rules from W to get *projective* sets $\Gamma_1, \ldots, \Gamma_n$ such that:

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Now let W' be the set of single-conclusion rules ($n \in \mathbb{N}$):

$$(\mathrm{W}'_n) \quad (\vec{p} \to \bot), ((\neg\neg p_1 \to p_1) \to q), \ldots, ((\neg\neg p_n \to p_n) \to q) \ / \ q.$$

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W' *is a basis for the admissible single-conclusion rules of* L*.*

The (W'_n) rules are not IPC-derivable but are derivable in, e.g., Gödel logic (IPC + $(p \to q) \lor (q \to p)$) and De Morgan logic (IPC + $\neg p \lor \neg\neg p$).

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Other Results

- The set of admissible rules of $IPC_{\to\neg}$ is PSPACE-complete.
- L is (hereditarily) structurally complete iff all L-frames are **almost Church-Rosser** (each finite set of non-maximal elements with a lower bound also has an upper bound).
- The implication-negation fragment of any intermediate logic with the disjunction property is not structurally complete.
- The set of admissible rules of $IPC_{\to\neg}$ has no finite basis.
- If L is strictly contained in classical logic, then L has finitary (not unitary) unification type.

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A Useful Admissible Rule

Proving the admissibility (e.g., by *elimination*) of a **cut rule** such as

$$\frac{\Gamma_1 \Rightarrow \varphi \quad \Gamma_2, \varphi \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

for a proof system can be crucial to obtain decidability, complexity, interpolation results etc. for a logic or class of algebras.

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Another Useful Rule

Similarly, proving the admissibility of a **density rule** such as

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where p does not occur in the conclusion.

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Rewriting Density

The corresponding first-order formula is

$$\forall q, r, s\, \exists p\, [\, 1 \leq q \vee (r \to p) \vee (p \to s) \;\;\Rightarrow\;\; 1 \leq q \vee (r \to s)\,].$$

Skolemizing, we obtain a quasiequation

$$1 \leq q \vee (r \to d(q,r,s)) \vee (d(q,r,s) \to s) \;\;\Rightarrow\;\; 1 \leq q \vee (r \to s)$$

and hence a rule

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An Application of Unboundedness?

- The variety $\mathcal{C}$ of **Komori's C algebras** (the algebras of the implicational fragment of Łukasiewicz logic) is generated by $\langle \mathbb{Z}^-, \to \rangle$ where $x \to y = \min(0, y - x)$, but this is hard to show…
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Admissibility of these rules for certain varieties of CRLs can also be shown using proof-theoretic methods.

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$$\exists p\, \forall q\, [\, 1 \leq p \to q\,] \qquad \text{and} \qquad \exists p\, \forall q\, [\, 1 \leq q \to p\,]$$

and, skolemizing, the rules

$$\emptyset \,/\, \bot \to q \qquad \text{and} \qquad \emptyset \,/\, q \to \top.$$

Admissibility of these rules for certain varieties of CRLs can also be shown using proof-theoretic methods.

Concluding Remarks

- Admissible rules play an interesting role in logic and algebra.
- Structural completeness and other admissibility issues have been addressed for IPC and its fragments.
- However, there remain many open questions for other non-classical logics…
- Admissible rules can be useful in establishing properties of logics or classes of algebras.

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