CSP on small templates

B. Bašić, J. Jovanović,
P. Marković, S. Moconja and P. Tanović

University of Novi Sad, Serbia

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Dedication

The work presented here is dedicated to the memory of Jarda Ježek.

Constraint Satisfaction Problem - definition

We are trying to determine whether there exists a homomorphism from an input relational structure B into a fixed relational structure A of the same finite similarity type.

This problem is denoted $CSP(A)$ and called the fixed-template constraint satisfaction problem.

In fact, we are trying to find the time complexity of the above question.

Clearly, it is in the class NP, as we can verify that a given mapping from B to A is a homomorphism in a time which is polynomial in the size of B.

On the other hand, we can easily find A for which $CSP(A)$ is NP-complete, for example:

- $A = \langle \{0, 1, 2\}; \neq \rangle$ (3-colorability)
- $A = \langle \{0, 1\}; \{0, 1\}^3 \setminus \{(0, 0, 0), (1, 1, 1)\} \rangle$ (Not-all-equal 3-satisfiability)

Dichotomy Conjecture

It is not hard to construct A such that $CSP(A)$ is in P, sometimes it is in even nicer complexity classes (if we use finer than polytime reductions), like NL and L.

In their major 1999 paper (written in 1993), T. Feder and M. Vardi stated the conjecture that

Dichotomy Conjecture

For all finite relational structures A, the problem $CSP(A)$ is either in P or NP-complete.

Their paper also makes several reductions of the above conjecture, as well as establishing its 'proximity' to the Millenium Problem 'P vs NP'.

Variants of the problem

Instead of the homomorphism between relational structures, we may look at an equivalent from logic

PP-satisfiability

Is an input formula which is a closed primitive positive formula true in a fixed finite relational structure A?

Or another equivalent which looks more computer science related:

Variable-value version

Given a fixed finite relational structure A (the elements of which are called values), if we input a finite set B (variables) and a set of ordered pairs $\{(\overrightarrow{x_1}, R_{i_1}), \ldots, (\overrightarrow{x_m}, R_{i_m})\}$ (constraints), where R_{i_j} are k_j-ary relations of the structure A of and $x_j \in B^{k_j}$, does there exist a mapping $f : B \to A$ such that for all constraints, $f(\overrightarrow{x_j}) \in R_{i_j}$?

Variants of the problem - harder versions

We may also allow A to be a finite structure in an infinite language, and then B must have only finitely relations from the language of A which are nonempty (so that our input is finite).

Algebraic version: A may be given by a set of operations on its base set (i. e. we fixed a finite algebra), and the (infinite) set of relations is the set of all compatible relations (i. e. subpowers) of this algebra.

This is a Galois connection between closed sets of relations under primitive-positive definitions (relational clones) and the sets of operations which contain all projections and are closed under compositions.

These two problems are also conjectured to be either tractable or NP-complete, and positive answer to one of those would imply a positive answer to the original dichotomy conjecture.

Also, there is a version of the problem where even A is allowed to be infinite, but I know very little about it. Except that ω-categoricity helps, since then the duality works as well as in the finite case.

Algebraic version

Bulatov, Jeavons and Krokhin proved that the problem $CSP(A)$ depends only on the clone which is compatible with the relations of A, that without changing complexity we can change A to some A' which has all unary one-element relations and that if the dual clone of A' contains no Taylor operations, then the problem $CSP(A')$ is NP-complete.

Algebraic Dichotomy Conjecture

For all finite relational structures A which contain all unary one-element relations, if there is a Taylor operation on the set A compatible with the relations of A, then the problem $CSP(A)$ is in P.

This, as well as the results about stronger versions of Dichotomy from the previous slide, have been verified for various conditions which imply having a Taylor operation.

Taylor operation equivalents

Maróti and McKenzie give a workable equivalent version of Taylor operations, called weak near-unanimity operations:

WNU definition

$w(x, x, \ldots, x) = x$ and
$w(x, x, \ldots, x, y) = w(x, x, \ldots, x, y, x) = \ldots = w(y, x, x, \ldots, x)$

The obviously definable derived binary operation from wnu will be denoted $x \circ_w y := w(x, x, \ldots, x, y)$.

There are other, stronger or different equivalent conditions for Taylor terms, like cyclic terms or the Siggers term, but we won't use them here.

Ancient history

In the year of the three birthdays, E. Post described the clone lattice on the two element set.

Schaefer was able to prove the Dichotomy Conjecture for $|A| = 2$ in 1978, 15 years before it was conjectured. Today it is a trivial consequence of the Post lattice, as we have algorithms for all small clones with Taylor operations on the 2-element set.

Recent history

The clone lattice on a finite set with more than 2 elements is of the size continuum, so Bulatov managed to extend Schaefer's result to three elements only in 2006.

His proof sparked a lot of other results, where Bulatov proved the Dichotomy Conjecture for all finite relational structures having all subsets as unary relations (submitted), and having all unary relations with at most three elements (announced).

It also inspired R. Mckenzie, M. Nickodemus and the speaker, later also M. Maróti, to work on some combinations of semilattices and Mal'cev operations. This work is still unfinished, but useful, as you will see.

We decided to try for 4-element case to see what new tricks we learn from doing it.

Reductions

Our strategy

We will classify all algebras $\langle\{0,1,2,3\};w\rangle$ with respect to the operation $\circ_w$ which w induces and then solve each.

Given that any binary operation with $x \circ_w x = x$ is realizable, we have only $4^{12} = 2^{24}$ cases to consider. That is just over 16 million. Piece of cake.

We better reduce the number of cases first. So we define a digraph:

Reduction digraph

If for all algebras $\langle\{0,1,2,3\};w\rangle$ such that $\circ_w = \circ$ there exists a weak nu term operation v of $\langle\{0,1,2,3\};w\rangle$ such that $\circ_v = *$ then we say that $\circ$ reduces to $*$ and write $\circ \to *$.

Reductions, old

So what reductions do we have? An old one, due to M. Maróti and R. McKenzie says

Theorem (Special wnu)

Let $\circ$ be a binary idempotent operation on an algebra A. Then $\circ \longrightarrow *$, where $x * y = x \circ (x \circ (\ldots x \circ (x \circ y) \ldots))$. Particularly, when we use $|A|!$ many $\circ$s, then $*$ satisfies $x * (x * y) = x * y$. We call wnu w for which $x \circ_w (x \circ_w y) = x \circ_w y$ a special *wnu.*

This leaves us with only $23^4 = 279841$ tables, as there are 23 options for each row (prove it).

But also we may take one table per isomorphism type, so we are down to 11891 groupoids only.

Reductions, new

We can prove that $\circ \to *$ when

- $x * y = (x \circ y) \circ (y \circ (x \circ y))$
- $x * y = (\dots(((x \circ y) \circ x) \circ x) \dots \circ x) \circ (\dots(((x \circ y) \circ y) \circ y) \dots \circ y)$
- $x * y = (x \circ y) \circ ((y \circ x) \circ y)$
- $x * y = (x \circ (y \circ x)) \circ ((x \circ y) \circ y)$
- If for all $a, b \in A$, $a \circ (b \circ a) \in \{a \circ b, b \circ a\}$, then

$$a * b = \begin{cases} (a \circ b) \circ (b \circ a) & \text{if} \quad a \circ (b \circ a) = a \circ b; \\ (b \circ a) \circ (a \circ b) & \text{if} \quad a \circ (b \circ a) = b \circ a. \end{cases}$$

- If $(x \circ y) \circ y = x \circ y$ for all $x, y \in A$ and for all $a, b \in A$, $(a \circ b) \circ a \in \{a \circ b, b \circ a\}$, then

$$a * b = \begin{cases} (a \circ b) \circ (b \circ a) & \text{if} \quad (a \circ b) \circ a = a \circ b; \\ (b \circ a) \circ (a \circ b) & \text{if} \quad (a \circ b) \circ a = b \circ a. \end{cases}$$

We are down to 746 cases now.

Multisorted CSP

A template $\mathcal{B}$ for the constraint satisfaction problem is a set of finite idempotent algebras of similar type closed under taking subalgebras, homomorphic images and retracts via idempotent unary polynomials.

Then for each variable we have a sort. Now the constraints are subuniverses of the product of sorts, instead of the power of a sort.

We were able to prove the following theorem:

Theorem

A multisorted CSP instance with each sort having at most three elements is tractable when there is a weak nu operation w_i on each sort (w_i is the interpretation of the syntactical operation w in the ith sort).

What are we trying to do?

Definition

When A is a sort in a multisorted CSP template T which is neither a subuniverse, nor a homomorphic image, nor a polynomial image of any other sort, we say that A can be eliminated if this multisorted CSP is not harder than multisorted CSP with template $T \setminus \{A\}$.

So our theorem from previous slide implies that if a four element sort can be eliminated then it REALLY can be eliminated (the single sorted version over this algebra converts to a multisorted instance over at most three element sorts).

Maróti's lemma

Lemma

Let $t(x, y)$ be an operation such that

- $t(x, t(x, y)) = t(x, y)$ *on* A,
- $(\forall x \in A)(\exists y \in A) t(x, y) \neq y$, *and*
- *The set* $S = \{y \in A : g_y(x) := t(x, y)$ *is a permutation*$\}$ *generates a proper subuniverse of* A,

then the sort A *can be eliminated.*

An idea in the proof: Take more coordinates while reducing the sizes of potatoes. If the original problem has a solution, then the new one has a solution. If the new problem has a solution, then it tells us how to reduce the original problem.

Things to remember

- Maróti's Lemma is very applicable as it just asks for a random binary term, and the first condition can be forced by iteration,
- If it fails, then some subset S is not contained in any subuniverse and
- It can be used iteratively.

We are down to only 130 cases now.

Consistency - fixed template

We want the following conditions:

- For all subsets I of the set of variables with at most k elements, we have precisely one constraint $R_I \leq A^I$.
- We have no constraints with more than k coordinates.
- for all $I \subset J$ and $|J| \leq k$, $R_I = (R_J) \restriction_I$.

We can get an equivalent problem with these properties in polytime in the fixed-template case.

Consistency - algebraic version

In the algebraic version we get a bit less, but enough for our purposes:
Fix a linear order on the variables.

- For all subsets I of the set of variables with at most 2 elements, we have precisely one constraint $R_I \leq A^I$.
- We have at most one constraint on subsets of variables with more than 2 variables.
- For each I such that there exists a constraint R_I, there exists a constraint for each J which is the intersection of I with an initial segment.
- For all $I \subset J$, $(R_J) \restriction_I \subseteq R_I$ and if $|I| \leq 2$, $R_I = (R_J) \restriction_I$.
- For all $i, j, k \in B$ and $a, b \in A$, if $(a, b) \in R_{\{i,j\}}$, then there exists a $c \in A$ so that $(a, c) \in R_{\{i,k\}}$ and $(b, c) \in R_{\{j,k\}}$.

We say A has *bounded width* when, for all consistent instances over A, if R_I is nonempty whenever R_I exists, then that instance has a solution.

Barto-Kozik result

L. Barto and M. Kozik proved that whenever A is a finite idempotent algebra which generates a congruence meet-semidistributive variety, then all CSP problems over A have bounded width.

Among numerous equivalent conditions for A generating a congruence meet-semidistributive variety, we select the one saying that there are no Abelian algebras in $HS(A)$.

Every Abelian algebra which has a weak nu operation is polynomially equivalent to a module over a finite ring.

It is now an easy linear algebra exercise to show that any special weak nu in an Abelian algebra must be a minority operation.

Applying bounded width

So we know that there must be a subuniverse of the $\circ$ groupoid and a congruence on it such that the factor of the subgroupoid is second projection. In all cases except when w is a full minority operation this implies that there is at least one nontrivial subuniverse. This can be used against Maróti's Lemma to derive a contradiction.

Another result which picks up a few cases is the GMM operation - we can use Dalmau's Gemeralized majority-minority algorithm whenever on each two-element subset the operation $\circ$ acts as one of the projections.

When we eliminate all tables which are guaranteed to be solvable by bounded width and gmm operation, we are left with 99 cases.

And after searching for more complicated terms than $x \circ y$ or $y \circ x$ for applying Maroti's Lemma we are at 71 cases.

Finally, when we let the program use Maróti's Lemma to derive partial tables, which can be reused in Maróti's Lemma and/or used to prove that there is no Abelian member of $HS(A)$, then we are down to 20 cases.

Neutral element

When we have a neutral element, Miklós is helpless!

The main problem he was trying to solve, the semilattice of Mal'cev algebras, fails to reduce by his lemma when there is a top element of the semilattice, consisting of an one-element algebra.

In other words, when there is a neutral element.

Therefore there is little wonder that the majority of the remaining 20 cases has a neutral element.

Least block algorithm

The algorithm invented by (some subset of) R. McKenzie and the speaker for dealing with linearly ordered semilattice of Mal'cev algebras.

The idea: Let there be a solution of the restriction of the problem to first k variables going always through the least class and no solution through the least class for the restriction of the problem to first $k+1$ variables. Which is why we need the linear order on coordinates in the consistency reduction. Then we prove that no solution for the full problem can go through the least block at $k+1$st variable and reduce the problem.

We use this for two blocks: neutral element above and all other elements below.

A little lemma

We need two things: That all elements but the neutral one form a subuniverse, but we get that by switching to a derived wnu operation with the same $\circ$.

And the other one is:

Lemma

Let A be a wnu algebra and $\circ$ have a neutral element 1 and all other elements form a subuniverse of A. Let there exist a term $t(x, y)$ such that $t(1, y) = 1$ iff $y = 1$ and $t(x, y) = f(x)$ when $x \neq 1$, then the least block algorithm works for A.

Of course, we are not guaranteed this term, but when no such term exists, we were always able to emulate it using polynomials.

Odds and ends

We are left with 7 sporadic cases. Six of them are done by polynomial reductions of the largest sort in a multisorted version of the problem, and using the fact that many sorts can be excluded via Maróti's Lemma.

And we are down to one case.

$\circ_w$	0	1	2	3
0	0	0	2	2
1	1	1	3	3
2	0	1	2	3
3	0	1	2	3

Final thoughts

What have we learned in general?

- Maróti's lemma is more general than it looks.
- Maróti's lemma and bounded width work well against each other.
- We have new ideas for semilattices of Malcev algebras which may help us reduce the remaining situation when there is a neutral element.
- If all else fails resort to polynomials to find a Bulatov-style reduction specific to the case.

Digraphs

For compleness sake, we should mention that there is also a result by Barto and Stanovský showing that for digraphs with all unary one-element relations and at most five vertices, the Dichotomy Conjecture holds.

They proved it using computer search. This search also generated a six-element example which failed to be bounded width, or gmm. They called their graph *The Sea-devil Digraph*.

We proved that the sea devil is tractable.