Quasi-subtractive varieties

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Joint work with Antonio Ledda, Francesco Paoli, Matthew Spinks

A very classical beginning

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- Gumm and Ursini developed a general theory of ideals in universal algebra. They identified classes of varieties for which ideals behave well.

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with 1 playing the role of 0 and filters replacing ideals. This will be our "official" notion of subtractivity. Most of the theory of subtractive varieties is due to Aglianò and Ursini.

1-permutability and 1-regularity

$\mathcal{V}$ is called 1-permutable iff for any algebra $\mathbf{A} \in \mathcal{V}$ and for any congruences θ, φ on $\mathbf{A}$, we have $1^{\mathbf{A}}/\theta \circ \varphi = 1^{\mathbf{A}}/\varphi \circ \theta$.

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$$1/\theta = 1/\varphi \text{ implies } \theta = \varphi$$

for any $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$, $\mathbf{A} \in \mathcal{V}$. Point-regular subtractive varieties are known as ideal-determined.

Two properties of subtractive varieties

Theorem

In every member **A** *of an ideal-determined variety* $\mathcal{V}$ *there is a lattice isomorphism between the lattice of congruences of* **A** *and the lattice of ideals (filters) in the sense of Gumm-Ursini. Moreover, these ideals (filters) coincide with deductive filters of the* 1*-assertional logic corresponding to* $\mathcal{V}$*.*

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Theorem

In every member **A** *of a subtractive variety* $\mathcal{V}$ *the lattice of ideals (filters) in the sense of Gumm-Ursini is modular.*

Above and beyond: three examples

- *Pseudointerior algebras*. Not subtractive, but have a manageable concept of *open filter* (distinct from Gumm-Ursini ideal/filter). In every pseudointerior algebra there is an isomorphism between the lattices of congruences and of open filters.

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- *Pseudointerior algebras*. Not subtractive, but have a manageable concept of *open filter* (distinct from Gumm-Ursini ideal/filter). In every pseudointerior algebra there is an isomorphism between the lattices of congruences and of open filters.
- *Residuated lattices*. Ideal determined, but there is another isomorphism: between congruences and deductive filters. This is not subsumed by the general results.
- *Quasi-MV algebras*. Neither subtractive nor 1-regular. Still, in every quasi-MV algebra the lattice of certain "good" congruences is isomorphic to the lattice of certain filter-like subsets.

More about quasi-MV algebras

A quasi-MV algebra is an algebra $\mathbf{A} = \langle A, \oplus, ', 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following equations:

A1. $x \oplus (y \oplus z) \approx (x \oplus z) \oplus y$
A2. $x'' \approx x$
A3. $x \oplus 1 \approx 1$
A4. $(x' \oplus y)' \oplus y \approx (y' \oplus x)' \oplus x$
A5. $(x \oplus 0)' \approx x' \oplus 0$
A6. $(x \oplus y) \oplus 0 \approx x \oplus y$
A7. $0' \approx 1$

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A quasi-MV algebra is an MV algebra if and only if it satisfies the equation $x \oplus 0 \approx x$. It is flat if and only if it satisfies the equation $x \oplus 0 \approx y \oplus 0$.

More about quasi-MV algebras

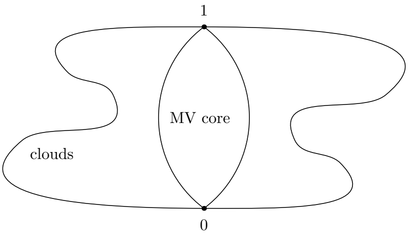

Figure: A typical quasi-MV algebra

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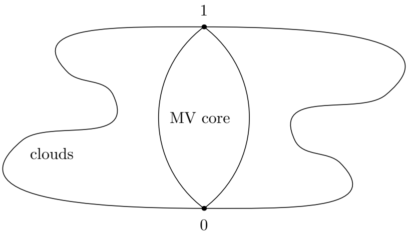

Figure: A typical quasi-MV algebra

Theorem (Ledda, Konig, Paoli, Giuntini)

The variety of quasi-MV algebras decomposes as a subdirect product of (the varieties of) MV algebras and flat algebras.

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$$\tau^{\mathbf{A}}/\theta = \{a \in \mathbf{A} \colon \delta_i^{\mathbf{A}}(a)\theta\epsilon_i^{\mathbf{A}}(a) \text{ for every } i \leq n\}.$$

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Appropriate notions of τ-permutability and τ-regularity can then be defined (and τ-permutability turns out to be equivalent to existence of some binary terms satisfying certain equations). So, relativising subtractivity to some τ looks like just what the doctor ordered. Our τ will be $\{\Box x \approx 1\}$.

Quasi-subtractive varieties

Definition (Quasi-subtractive varieties)

A variety $\mathcal{V}$ whose type ν includes a nullary term 1 and a unary term $\Box$ is called quasi-subtractive with respect to 1 and $\Box$, if there is a binary term $x \to y$, such that $\mathcal{V}$ satisfies the equations

Q1. $\Box x \to x \approx 1$

Q2. $1 \to x \approx \Box x$

Q3. $\Box(x \to y) \approx x \to y$

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Conditions Q1 and Q2 are jointly equivalent to being τ-permutable, for $\tau = \{\Box x \approx 1\}$. Conditions Q3 and Q4 are less straightforward to justify, but without them the lattice of open filters would not be modular.

Two examples and a caveat

Example (Subtractive varieties)

Every subtractive variety $\mathcal{V}$ is quasi-subtractive: it suffices to take as arrow the term witnessing subtractivity for $\mathcal{V}$, and as box the identity term.

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Let $\mathcal{V}$ be any pointed variety, i.e., a variety whose type includes a constant 1. Defining $\Box x = 1 = x \to y$ it is immediately verified that $\mathcal{V}$ is quasi-subtractive with the above witness terms.

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> *Life is like a sewer. What you get out of it depends on what you put into it. — Tom Lehrer*

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Quasi-subtractivity deals with varieties where $1/\theta$ may not behave. Subsets that behave can be bigger than $1/\theta$. We will call them open filters.

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Good behaviour of open filters can be expressed by certain "relativised Mal'cev conditions" corresponding to them.

Open filters

Let $\mathcal{V}$ be quasi-subtractive, and $\mathbf{A} \in \mathcal{V}$.

Definition

An open filter term in the variables $\mathbf{x}$ is is an $n+m$-ary term $p(\mathbf{x}, \mathbf{y})$ such that $\Box\mathbf{x} \approx 1$ implies $\Box p(\mathbf{x}, \mathbf{y}) \approx 1$

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Definition

An open filter of $\mathbf{A}$ is a subset $F \subseteq A$ such that:

1. if p is an open filter term, and $\mathbf{a} \in F, \mathbf{b} \in A$, then $p(\mathbf{a}, \mathbf{b}) \in F$
2. $a \in F$ iff $\Box a \in F$

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Theorem

Let $\mathcal{V}$ be a quasi-subtractive variety, and $\mathbf{A} \in \mathcal{V}$. The set of open filters of $\mathbf{A}$ under the operations of intersection and $\uparrow\Gamma$ of union, forms an algebraic modular lattice.

Open and flat subvarieties

Let $\mathcal{V}$ be quasi-subtractive. The subvariety $\mathcal{V}_O$ of $\mathcal{V}$ defined by $\Box x \approx x$ we call open. Any subvariety whose intersection with $\mathcal{V}_O$ is trivial, we call flat.

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Lemma

Let $\mathcal{V}_F \subseteq \mathcal{V}$ be flat. Then, there exists a unary term $\boxtimes x$ such that $\mathcal{V}_O \models \boxtimes x \approx x$ and $\mathcal{V}_F \models \boxtimes x \approx 1$.

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Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be varieties. We write $\mathcal{V}_1 \times_s \mathcal{V}_2$ for the class $\{\mathbf{A} \hookrightarrow_s \mathbf{B}_1 \times \mathbf{B}_2 \colon \mathbf{B}_1 \in \mathcal{V}_1 \text{ and } \mathbf{B}_2 \in \mathcal{V}_2\}$.

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Theorem

If $\boxtimes$ commutes with all operations not preserving $\{1\}$ on all algebras in $\mathcal{V}_O \cup \mathcal{V}_F$, then $\mathcal{V}_O \vee \mathcal{V}_F = \mathcal{V}_O \times_s \mathcal{V}_F$.

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Varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ are independent if there is a binary term $x \star y$ such that $\mathcal{V}_1 \models x \star y = x$ and $\mathcal{V}_1 \models x \star y = y$.

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Theorem

Let $\mathcal{V}$ be a variety of groups. The following are equivalent.

1. *$\mathcal{V}$ satisfies the identities $x^{k(k-1)} = e$ and $(xy)^{1-k}(zu)^k = x^{1-k}z^k y^{1-k}u^k$ for some $k > 1$.*
2. *$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$, for independent varieties $\mathcal{V}_1$ and $\mathcal{V}_2$.*

Some known results

Let $\mathcal{V}$ be a quasi-subtractive variety with open and flat subvarieties $\mathcal{V}_O$ and $\mathcal{V}_F$ such that, for some binary term $x \circ y$ and unary term $\boxdot$, the following hold:

1. $\mathcal{V}_O \models \boxdot x \approx 1, \quad x \circ 1 \approx x$
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Some known direct decomposition theorems become corollaries. Such are the decomposition theorems for certain varieties of residuated lattices, due to Jónsson-Tsinakis and Galatos-Tsinakis, or for *sircomonoids* due to Raftery-Van Alten.

Transfer of CEP and AP

Theorem

Let $\mathcal{V}_O$ and $\mathcal{V}_F$ be an open and flat subvarieties of a quasi-subtractive variety $\mathcal{V}$. Then, $\mathcal{V}_O \vee \mathcal{V}_F$ has CEP if and only if both $\mathcal{V}_O$ and $\mathcal{V}_F$ have CEP.

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Let $\mathcal{V}_O$ and $\mathcal{V}_F$ be an open and flat subvarieties of a quasi-subtractive variety $\mathcal{V}$. Then,

1. *$\mathcal{V}_O \vee \mathcal{V}_F$ has AP iff $\mathcal{V}_O$ and $\mathcal{V}_F$ have AP,*
2. *$\mathcal{V}_O \vee \mathcal{V}_F$ has SAP iff $\mathcal{V}_O$ and $\mathcal{V}_F$ have SAP.*

Open contractions

For any term $t(\mathbf{x})$, we define its open translation $t^{\Box}$ inductively:

- $x^{\Box} = x$, for a variable x,
- $o^{\Box}(t_1, \ldots, t_k) = \Box o(t_1^{\Box}, \ldots, t_k^{\Box})$, for a k-ary basic operation o and terms $t_1, \ldots, t_k$.

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On $A^{\Box} = \{a \in A \colon \Box a = a\}$, we define operations, putting $(o^{\Box})_{o \in O}$, where O is the set of all basic operations in the type. Then $\mathbf{A}^{\Box}$ is the algebra $\langle A^{\Box}, (o^{\Box})_{o \in O} \rangle$.

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Functoriality

Lemma

Let $h\colon \mathbf{A} \to \mathbf{B}$ be a homomorphism. Then, $h|_{A^\Box}\colon \mathbf{A}^\Box \to \mathbf{B}^\Box$ is a homomorphism, and the diagram

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{\quad h \quad} & \mathbf{B} \\ {\scriptstyle \Box}\downarrow & & \downarrow{\scriptstyle \Box} \\ \mathbf{A}^\Box & \xrightarrow{\quad h|_{A^\Box} \quad} & \mathbf{B}^\Box \end{array}$$

commutes. In particular, if θ is a congruence on $\mathbf{A}$, then $\theta|_{\mathbf{A}^\Box}$ is a congruence on $\mathbf{A}^\Box$.

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Suppose $\mathbf{A} \in \mathcal{V}$. In general, $\mathbf{A}^\Box$ may not belong to $\mathcal{V}$. Things begin to improve if the open translation preserves some structure.

Divide . . .

Let $\mathcal{V}^{\square} = \{\mathbf{A}^{\square} \colon \mathbf{A} \in \mathcal{V}\}$. An open contraction $\mathbf{A}^{\square}$ is:

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Let $\mathcal{V}^{\Box} = \{\mathbf{A}^{\Box} : \mathbf{A} \in \mathcal{V}\}$. An open contraction $\mathbf{A}^{\Box}$ is:

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- invertible, if for every algebra $\mathbf{A} \in \mathcal{V}$ and every congruence φ on $\mathbf{A}^{\Box}$ there is a congruence θ on $\mathbf{A}$ such that $\varphi = \theta|_{\mathbf{A}^{\Box}}$.

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Let $\mathcal{V}^{\Box} = \{\mathbf{A}^{\Box} \colon \mathbf{A} \in \mathcal{V}\}$. An open contraction $\mathbf{A}^{\Box}$ is:

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- invertible, if for every algebra $\mathbf{A} \in \mathcal{V}$ and every congruence φ on $\mathbf{A}^{\Box}$ there is a congruence θ on $\mathbf{A}$ such that $\varphi = \theta|_{\mathbf{A}^{\Box}}$.

Some scattered facts on these:

- There exist non-smooth open translations.
- The notions are pairwise independent (e.g., translation from ℓ-groups to their negative cones is invertible, but not contractive; translation from residuated lattices to negative cones is contractive but not invertible).
- A subvariety of a contractive variety may fail to be contractive.

. . . et impera

Theorem

Let $\mathbf{A}$ be quasi-subtractive with witness terms $\Box$ and $\to$. If $\mathbf{A}^{\Box}$ is smooth, then $\mathbf{A}^{\Box}$ is subtractive with witness term $\to^{\Box}$.

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Let $\mathcal{V}$ *be smooth and contractive. Then, the class* $\mathcal{V}^{\Box}$ *is a variety and it coincides with* $\mathcal{V}_O$*, the open subvariety of* $\mathcal{V}$*.*

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Theorem

Let $\mathcal{V}$ *be smooth and invertible and* $\mathbf{A} \in \mathcal{V}$. *Suppose* F *is a* $\mathcal{V}^{\Box}$*-open filter on* $\mathbf{A}^{\Box}$. *Then,* $\uparrow F$ *is a* $\mathcal{V}$*-open filter on* $\mathbf{A}$.

Bases for contractive varieties

An identity $t(\mathbf{x}) \approx s(\mathbf{x})$ will be called stable if it survives open translation, that is, if

- $\mathcal{V} \models t(\mathbf{x}) \approx s(\mathbf{x})$, and
- $\mathcal{V} \models t^{\Box}(\Box\mathbf{x}) \approx s^{\Box}(\Box\mathbf{x})$,

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where $t^{\Box}$ and $s^{\Box}$ are the respective open translations of t and s.

Theorem

Let $\mathcal{V}$ be quasi-subtractive. Then the following are equivalent:

1. *$\mathcal{V}$ is contractive;*
2. *$\mathcal{V}$ has a basis of stable identities;*
3. *every basis of $\mathcal{V}$ consists of stable identities;*
4. *the equational theory of $\mathcal{V}$ consists of stable identities.*

Stable expansions

For a quasi-subtractive $\mathcal{V}$, we define its stable expansion $\mathcal{V}_s$ to be the class of models of the stable part of the equational theory of $\mathcal{V}$. Namely, we put $\mathcal{V}_S = \mathrm{Mod}\{Eq(\mathcal{V}) \cap Eq(\mathcal{V}^\Box)\}$.

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For any quasi-subtractive smooth variety $\mathcal{V}$, we have $(\mathcal{V}_S)^\Box = \mathcal{V}^\Box$. Thus, the stable expansion $\mathcal{V}_S$ of $\mathcal{V}$ is contractive.

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Theorem

Let $\mathcal{V}$ be quasi-subtractive with witness terms 1, $\Box$ and $\to$, which are smooth. Then, for $\mathcal{W} = \mathcal{V} \vee \mathcal{V}^{\Box}$ we have:

- *$\mathcal{W}$ is contractive.*
- *$\mathcal{W}$ is quasi-subtractive with the same witness terms.*
- *$\mathcal{W}$ is precisely the class of models of stable identities of $\mathcal{V}$.*
- $\mathcal{W}^{\Box} = \mathcal{V}^{\Box} = \mathcal{W}_O$.

Examples of open contractions

Example

Let $\mathcal{V}$ be the variety of quasi-MV algebras, and $\Box x = x \oplus 0$, $x \to y = \neg x \oplus y$. Then, $\mathcal{V}^{\Box} \subseteq \mathcal{V}$ is the variety of MV algebras.

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Let $\mathcal{V}$ be the variety of residuated lattices, and $\Box x = x \wedge 1$, $x \to y = x \backslash y$. Then $\mathcal{V}^{\Box} \subseteq \mathcal{V}$ is the variety of integral residuated lattices, and the translation is known as the negative cone contraction.

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Example

Let $\mathcal{V}$ be the variety of ℓ-groups, and $\Box x = x \wedge 1$, $x \rightarrow y = x^{-1}y$. Then $\mathcal{V}^{\Box}$ is the variety of negative cones of ℓ-groups, and $\mathcal{V} \vee \mathcal{V}^{\Box} = \mathcal{V} \times \mathcal{V}^{\Box}$.

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Let $\mathcal{V}$ be the variety of pseudointerior algebras, $\Box x = x^\circ$ and $x \to y = (x\backslash y)^\circ$, where $^\circ$ is the pseudointerior operation and $x\backslash y$ is the "pseudoresiduation". Then $\mathcal{V}^\Box \subseteq \mathcal{V}$ is the variety of Brouwerian semilattices.

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Example

Any variety $\mathcal{C}$ of Boolean algebras with conjugate operators (of finite type) directly decomposes as $\mathcal{C}_1 \times \mathcal{C}_2$, with

- $\mathcal{C}_1 \models \boxtimes 0 \approx 1$,
- $\mathcal{C}_2 \models \boxtimes 0 \approx 0$,

where $\boxtimes$ is the *master modality*.

Open contractions as translations between logics

Example

Let $\mathcal{V}$ be the variety of interior algebras and $\Box x$ be the interior operator. Then $\mathcal{V}^{\Box}$ is the variety of Heyting algebras, and the open translation is the usual Gödel translation.

Open contractions as translations between logics

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Let $\mathcal{V}$ be the variety of interior algebras and $\Box x$ be the interior operator. Then $\mathcal{V}^{\Box}$ is the variety of Heyting algebras, and the open translation is the usual Gödel translation.

Example

Let $\mathcal{V}$ be the variety of Heyting algebras and $\Box x = \neg\neg x$. Then $\mathcal{V}^{\Box} \subseteq \mathcal{V}$ is the variety of Boolean algebras, and the open translation is the usual Glivenko translation.

Whither must I wander?

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Commutator for open filters.

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Open contractions and translations between logics. Look at smoothness, contractivity and invertibility in this context.

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Characterise these quasi-subtractive $\mathcal{V}$ that decompose as $\mathcal{V} = \mathcal{V}_O \times_s \mathcal{V}_F$.

Problem

Have a closer look at relativised Mal'cev conditions.

Final advertisements

- T. K., F. Paoli, M. Spinks, Quasi-subtractive varieties, *Journal of Symbolic Logic*, forthcoming.
- T. K., F. Paoli, Joins and subdirect products of varieties, *Algebra Universalis*, forthcoming.

Sequels:

- F. Paoli, Wednesday, 3pm.
- A. Ledda, Wednesday, 4:30pm