Growth Functions of Finite Algebras

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Answer: $d_{\mathbf{A}}(n) := \min\{g : \mathbf{A}^n \text{ has a generating set of size } g\}$.

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The "Few Subpowers" paper (BIMMVW) proves that **A** has polynomially generated subpowers iff **A** has a cube term.

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5. (M. Quick, N. Ruškuc, 2010) If $\mathbf{A}$ is a finite ring, module, k-algebra or Lie algebra, then $d_{\mathbf{A}}$ is logarithmic if $\mathbf{A}$ is perfect and linear otherwise.

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3. If $\mathbf{A}$ is finite, then $\lceil \log_{|A|}(n) \rceil \leq d_{\mathbf{A}}(n) \leq |A|^n$.

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Write $\mathbf{A} \models F(1,x,2) \approx x$ *and* $\mathbf{A} \models F(x,y,3) \approx x$ *as a matrix equation:*

$$\mathbf{A} \models F\begin{pmatrix} 1 & x & 2 \\ x & y & 3 \end{pmatrix} \approx \begin{pmatrix} x \\ x \end{pmatrix}$$

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F is a p-pointed, m-ary, k-cube term for $\mathbf{A}$ if $\mathbf{A} \models F(M) \approx \begin{pmatrix} x \\ x \end{pmatrix}$ where M is a $k \times m$-matrix consisting of p constants and some variables, and every column of M contains a symbol different from x.

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Theorem

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Theorem

If Σ is a set of basic identities that entails the existence a pointed cube term, then any finite algebra realizing Σ has growth function bounded by a polynomial.

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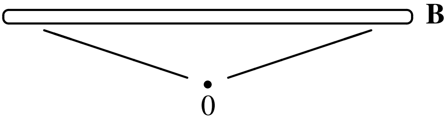

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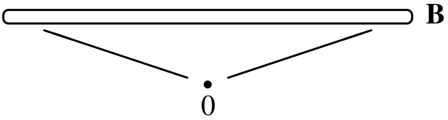

Stage 2: Show how to construct models of Σ of every cardinal larger than the number of constants in Σ.

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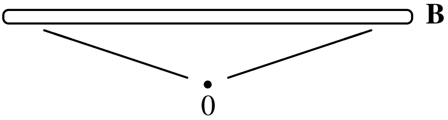

Stage 2: Show how to construct models of Σ of every cardinal larger than the number of constants in Σ.

Stage 3: Show how to merge a sufficiently large model from Stage 1 with a sufficently large model from Stage 2 to form $\mathbf{A}$ with $d_{\mathbf{A}} = d_{\mathbf{B}}$.

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Idea: Split coordinates of $[\mathbf{a}] \in \mathbf{A}^n$ *into* k *equal size segments,* $\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$. *Then*

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shows how to generate $[\mathbf{a}]$ *from more "highly processed" tuples.*

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Must show: there are only polynomially many fully processed tuples.

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At present, the only growth rates we know are logarithmic, polynomial or exponential.

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2. If finite **A** has a polynomial $B(x, y)$ with a 2-sided unit ($B(1, x) \approx x \approx B(x, 1)$), must the growth rate of **A** be linear or logarithmic?
3. Is there a nice description of finite algebras with polynomially generated powers?